

# SPECTRUM AND INDEX OF TWO-SIDED ALLEN–CAHN MINIMAL HYPERSURFACES

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**ABSTRACT.** The combined work of Guaraco, Hutchinson, Tonegawa and Wickramasekera has recently produced a new proof of the classical theorem that any closed Riemannian manifold of dimension  $n + 1 \geq 3$  contains a minimal hypersurface with a singular set of Hausdorff dimension at most  $n - 7$ . This proof avoids the Almgren–Pitts geometric min-max procedure for the area functional that was instrumental in the original proof, and is instead based on a considerably simpler PDE min-max construction of critical points of the Allen–Cahn functional. Here we prove a spectral lower bound for the hypersurfaces arising from this construction. This directly implies an upper bound for the Morse index of the hypersurface in terms of the indices of the critical points, provided it is two-sided. In particular, two-sided hypersurfaces arising from Guaraco’s construction have Morse index at most 1. Finally, we point out by an elementary inductive argument how the regularity of the hypersurface follows from the corresponding result in the stable case.

## 1. INTRODUCTION

A classical theorem, due to the combined work of Almgren, Pitts and Schoen–Simon, asserts that for  $n \geq 2$ , every  $(n + 1)$ -dimensional closed Riemannian manifold  $M$  contains a minimal hypersurface smoothly embedded away from a closed singular set of Hausdorff dimension at most  $n - 7$ . The original proof of this theorem is based on a highly non-trivial geometric min-max construction due to Pitts [Pit81], which extended earlier work of Almgren [Alm65]. This construction is carried out directly for the area functional on the space of hypersurfaces equipped with an appropriate weak topology, and it yields in the first instance a critical point of area satisfying a certain almost-minimizing property. This property is central to the rest of the argument, and allows to deduce regularity of the min-max hypersurface from compactness of the space of uniformly area-bounded stable minimal hypersurfaces with singular sets of dimension at most  $n - 7$ , a result proved for  $2 \leq n \leq 5$  by Schoen–Simon–Yau [SSY75] and extended to arbitrary  $n \geq 2$  by Schoen–Simon [SS81]. (The Almgren–Pitts min-max construction has recently been streamlined by De Lellis and Tasnady [DLT13] giving a shorter proof. However, their argument still follows Pitts’ closely and is in particular based on carrying out the min-max procedure directly for the area functional on hypersurfaces.)

In recent years an alternative approach to this theorem has been developed, whose philosophy is to push the regularity theory to its limit in order to gain substantial simplicity on the existence part. Specifically, this approach differs from the original one in two key aspects: first, it is based on

a strictly PDE-theoretic min-max construction that replaces the Almgren–Pitts geometric construction; second, for the regularity conclusions, it relies on a sharpening of the Schoen–Simon compactness theory for stable minimal hypersurfaces. The idea in this approach is to construct a minimal hypersurface as the limit-interface associated with a sequence of solutions  $u = u_i$  to the Allen–Cahn equation

$$(1.1) \quad \Delta u - \epsilon_i^{-2} W'(u) = 0$$

on the ambient space  $M$ , where  $W : \mathbb{R} \rightarrow \mathbb{R}$  is a fixed double-well potential with precisely two minima at  $\pm 1$  with  $W(\pm 1) = 0$ . Roughly speaking, if the  $u_i$  solve (1.1) and satisfy appropriate bounds, the level sets  $\{u_i = s\}$  for  $s \in (-1, 1)$  converge as  $\epsilon_i \rightarrow 0^+$  to a stationary codimension 1 integral varifold  $V$ . This fact was rigorously established by Hutchinson–Tonegawa [HT00], using in part methods inspired by the earlier work of Ilmanen in the parabolic setting [Ilm93]. Note that  $u_i$  solves (1.1) if and only if it is a critical point of the Allen–Cahn functional

$$(1.2) \quad E_{\epsilon_i}(u) = \int_U \epsilon_i \frac{|\nabla u|^2}{2} + \frac{W(u_i)}{\epsilon_i}.$$

If the solutions  $u_i$  are additionally assumed stable with respect to  $E_{\epsilon_i}$ , then Tonegawa and Wickramasekera proved that the resulting varifold  $V$  is supported on a hypersurface smoothly embedded away from a closed singular set of Hausdorff dimension at most  $n - 7$ , and moreover that its regular part  $\text{reg } V$  is stable with respect to the area functional [Ton05, TW12]. Their proof of this regularity result uses the regularity and compactness theory for stable codimension 1 integral varifolds developed by Wickramasekera [Wic14] sharpening the Schoen–Simon theory.

Stability of  $u_i$  means that the second variation of the Allen–Cahn functional  $E_{\epsilon_i}$  with respect to  $H^1(M)$  is a non-negative quadratic form. More generally the index  $\text{ind } u_i$  denotes the number of strictly negative eigenvalues of the elliptic operator  $L_i = \Delta - \epsilon_i^{-2} W''(u_i)$ , so that  $u_i$  is stable if and only if  $\text{ind } u_i = 0$ . Using min-max methods for semi-linear equations, Guaraco [Gua15] recently gave a simple, elegant construction of a solution  $u_i$  to (1.1) with  $\text{ind } u_i \leq 1$  and  $\|u_i\|_{L^\infty} \leq 1$ , and such that as  $\epsilon_i \rightarrow 0$ , the energies  $E_{\epsilon_i}(u_i)$  are bounded above and away from 0. The lower energy bound guarantees that the resulting limit varifold  $V$  is non-trivial. Since  $\text{ind } u_i \leq 1$ ,  $u_i$  must be stable in at least one of every pair of disjoint open subsets of  $M$ . This elementary observation, originally due to Pitts in the context of minimal surfaces, together with a tangent cone analysis in low dimensions, allowed Guaraco [Gua15] to deduce the regularity of  $V$  from the results of [TW12]. Here we show that the index bound persists in the limit provided  $V$  has trivial normal bundle. We also point out that the regularity follows in all dimension from the corresponding result in the stable case via an inductive argument that avoids the tangent cone analysis used in [Gua15].

**Corollary.** *Let  $M$  be a closed Riemannian manifold of dimension  $n + 1 \geq 3$ . Let  $V$  be the integral varifold arising as the limit-interface of the sequence  $(u_i)$  of solutions to (1.1) constructed in [Gua15]. Then  $\dim_{\mathcal{H}} \text{sing } V \leq n - 7$ . If  $\text{reg } V$  is two-sided, then its Morse index with respect to the area functional satisfies  $\text{ind}_{\mathcal{H}^n} \text{reg } V \leq 1$ .*

In min-max theory, one generally expects that the Morse index of the constructed critical point is no greater than the number of parameters used in the construction. The above shows this for two-sided hypersurfaces arising from Guaraco’s 1-parameter construction.

In fact, we prove a lower bound for  $(\lambda_p)$ , the spectrum of the elliptic operator  $L_V = \Delta_V + |A|^2 + \text{Ric}_M(\nu, \nu)$ —the *scalar Jacobi operator*—in terms of  $(\lambda_p^i)$ , the spectra of the operators  $(L_i)$ .

**Theorem.** *Let  $M$  be a closed Riemannian manifold of dimension  $n + 1 \geq 3$ . Let  $V$  be the integral varifold arising from a sequence  $(u_i)$  of solutions to (1.1) with  $\text{ind } u_i \leq k$  for some  $k \in \mathbb{N}$ . Then  $\dim_{\mathcal{H}} \text{sing } V \leq n - 7$  and*

- (a)  $\lambda_p(W) \geq \limsup_{i \rightarrow \infty} \lambda_p^i(W)$  for all  $W \subset\subset M \setminus \text{sing } V$ ,
- (b)  $\text{ind}_{\mathcal{H}^n} C \leq k$  for every two-sided connected component  $C \subset \text{reg } V$ .

*Remark.* The spectral lower bound of (a) also holds if the assumptions on the  $u_i$  are weakened in a spirit similar to [ACS15], that is if instead of an index upper bound one assumes that for some  $k \in \mathbb{N}$  there is  $\mu \in \mathbb{R}$  such that  $\lambda_k^i \geq \mu$  for all  $i$ . (Note that the index bound  $\text{ind } u_i \leq k$  is equivalent to  $\lambda_{k+1}^i \geq 0$ .)

For the minimal hypersurfaces obtained by a direct min-max procedure for the area functional on the space of hypersurfaces (as in the Almgren–Pitts construction), index bounds have recently been established by Marques and Neves [MN15]. Both the Almgren–Pitts existence proof and the Marques–Neves proof of the index bounds are rather technically involved; in particular, the min-max construction in this setting has to be carried out in a bare-handed fashion in the absence of anything like a Hilbert space structure. In contrast, in the approach via the Allen–Cahn functional, Guaraco’s existence proof is strikingly simple, and our proofs for the spectral bound and the regularity of  $V$  are entirely elementary bar the fact that they rely on the highly non-trivial sharpening of the Schoen–Simon regularity theory for stable hypersurfaces as in [Wic14].

**Outline of the paper.** In Section 2 we briefly expose notions from the theory of varifolds, set the context for the rest of the paper and give the statements of the main result and its corollaries. Their proof requires a number of preliminary results, which are contained in Section 3. The proof of the main result Theorem A is in Section 4, and is split into two parts: in the first part we prove the spectral lower bound by an inductive argument on  $\text{ind } u_i$ ; this immediately also implies the index upper bound. The proof of  $\dim_{\mathcal{H}} \text{sing } V \leq n - 7$  is given in the second part, and uses a similar inductive argument. There are two appendices: Appendix A contains two elementary lemmas from measure theory that are used repeatedly in Section 3. Appendix B gives a proof of Proposition 3.6, which is a straight-forward adaptation of the argument used in [Ton05] for the stable case.

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## 2. VARIFOLDS, STABILITY AND STATEMENT OF MAIN THEOREM

The setting is as follows:  $(M^{n+1}, g)$  is a closed (that is, compact without boundary) Riemannian manifold of dimension  $n + 1 \geq 3$ , and  $U \subset M$  is an arbitrary open subset, possibly equal to  $M$  itself.

**2.1. Varifolds: basic definitions.** An  $n$ -dimensional *varifold* in  $U$  is a Radon measure on the Grassmannian  $G_n(U) = \{(p, E) \mid p \in U, E \subset T_p M, \dim E = n\}$ —the set of  $n$ -planes over points in  $U$ . Varifolds can be thought of as ‘generalized surfaces’ with a possibly large set of singularities and multiple sheets.

An important subclass are the *integral varifolds*, which correspond to a pair  $(\Sigma, \theta)$  of a countably  $n$ -rectifiable set  $\Sigma \subset U$  and a Borel-measurable function  $\theta \in L^1_{\text{loc}}(\Sigma, \mathbb{N})$  via:

$$(2.1) \quad V_{\Sigma, \theta}(\phi) = \int_U \phi(x, T_x \Sigma) \theta(x) d\mathcal{H}^n(x) \quad \text{for all } \phi \in C_c(G_n(U)),$$

where  $T_x \Sigma$  is the  $\mathcal{H}^n$ -a.e. defined tangent space to  $\Sigma$ . The function  $\theta$  is called the *multiplicity function*, and can be thought of as the number of ‘sheets’ of the varifold. A sequence  $(V^i)$  *converges as varifolds* to  $V$  if they converge weakly as Radon measures on  $G_n(U)$ , i.e. if

$$(2.2) \quad \int_{G_n(U)} \phi dV^i \rightarrow \int_{G_n(U)} \phi dV \quad \text{for all } \phi \in C_c(G_n(U)).$$

If  $V$  is an  $n$ -varifold in  $U$ , we define its *weight measure*  $\|V\|$  by

$$(2.3) \quad \|V\|(\phi) = \int_U \phi(x) dV(x, S) \quad \text{for all } \phi \in C_c(U).$$

This measure can be thought of as a generalisation of the notion of ‘area’ for embedded submanifolds.

Consider an arbitrary vector field  $X \in C_c^1(U, TM)$  with well-defined flow  $(\Phi_t)$  at least for small time. We can deform  $V$  in the direction of  $X$  by pushing it forward via its flow, that is for all  $\phi \in C_c(G_n(U))$

$$(2.4) \quad (\Phi_t)_* V(\phi) = \int_{G_n(U)} \phi(\Phi_t(x), d\Phi_t(x) \cdot S) J\Phi_t(x) dV(x, S)$$

where  $d_x \Phi_t$  is the derivative of  $\Phi_t$  at  $x \in U$ , and  $J\Phi_t(x) = \det(d_x \Phi_t \circ d_x \Phi_t^*)^{\frac{1}{2}}$ .

Differentiating the corresponding weight measures  $\|(\Phi_t)_* V\|$  yields the *first variation* of  $V$ :

$$(2.5) \quad \delta V(X) = \left. \frac{d}{dt} \right|_{t=0} \|(\Phi_t)_* V\|(U).$$

When  $\delta V(X) = 0$  for all vector fields  $X \in C_c^1(U, TM)$ , we say that  $V$  is *stationary*. This corresponds to ‘minimality’ for smooth hypersurfaces.

By definition, the *regular part* of  $V$  is the set of points  $x \in U \cap \text{spt}\|V\|$  such that in a neighbourhood of  $x$ ,  $\text{spt}\|V\|$  is smoothly embedded in  $M$ . Its complement is the *singular part* of  $V$ , denoted  $\text{sing } V := U \cap \text{spt}\|V\| \setminus \text{reg } V$ . For a stationary integral varifold  $V$ , the Allard regularity theorem implies that  $\text{reg } V$  is a dense subset of  $U \cap \text{spt}\|V\|$  [Sim84, Ch. 5].

**2.2. Stability and the scalar Jacobi operator.** Throughout this section  $V$  will be a stationary integral  $n$ -varifold in  $U \subset M$ .

**Definition 2.1.** An integral varifold  $V$  is called *two-sided* if  $\text{reg } V$  is two-sided, that is if  $NV := N(\text{reg } V)$  admits a continuous non-vanishing section. When this fails,  $V$  is called *one-sided*.

*Remark 2.2.* When the ambient manifold  $M$  (or indeed the subset  $U \subset M$ ) is orientable, then  $\text{reg } V$  is two-sided if and only if it is orientable.

Suppose that  $V$  is two-sided, and fix a unit normal vector field  $N \in C^1(NV)$ , so that every function  $\phi \in C_c^1(\text{reg } V)$  corresponds to a section  $\phi N \in C_c^1(NV)$ . After extending the vector field  $\phi N$  to  $C_c^1(U, TM)$ —the chosen extension will not matter for our purposes—we can deform  $\text{reg } V$  with respect to its flow  $(\Phi_t)$ . As  $V$  is stationary, the first variation  $\delta V(\phi N) = 0$ . A routine calculation, the details of which can be found for instance in [Sim84, Sec. 9] shows that the second variation satisfies

$$(2.6) \quad \begin{aligned} \delta^2 V(\phi N) &= \left. \frac{d^2}{dt^2} \right|_{t=0} \|(\Phi_t)_* V\|(U) \\ &= \int |\nabla_V \phi|^2 - (|A|^2 + \text{Ric}_M(N, N))\phi^2 \, d\|V\|, \end{aligned}$$

where  $\nabla_V$  is the Levi-Civita connection on  $\text{reg } V$ ,  $A$  is the second fundamental form of  $\text{reg } V \subset M$ , and  $\text{Ric}_M$  is the Ricci curvature tensor on  $M$ .

The expression on the right-hand side can be defined for one-sided  $V$  by replacing  $N$  by an arbitrary measurable unit section  $\nu : \text{reg } V \rightarrow NV$ , but it loses its interpretation in terms of the second variation of the area. As the Ricci curvature term does not depend on the sign of  $\nu$ , the expression remains well-defined and smooth.

**Definition 2.3** (Scalar second variation). The *scalar second variation* of a stationary integral varifold  $V$  is the quadratic form  $B_V$  defined for  $\phi \in C_c^2(\text{reg } V)$  by

$$(2.7) \quad B_V(\phi, \phi) = \int_{\text{reg } V} |\nabla_V \phi|^2 - (|A|^2 + \text{Ric}_M(\nu, \nu))\phi^2 \, d\|V\|.$$

*Remark 2.4.* One can consider  $\text{reg } V$  as a stationary integral varifold in its own right by identifying it with the corresponding varifold with constant multiplicity 1. Its scalar second variation  $B_{\text{reg } V}$  differs from  $B_V$  in that the integral is with respect to  $d\mathcal{H}^n$  instead of  $d\|V\|$ ; we will briefly use this in Section 3.2.

*Remark 2.5.* When  $V$  is one-sided, the second variation of its area has to be measured with respect to variations in  $C_c^1(NV)$ —we refer to [CM11, Sec. 1.8] for further information on this. We called  $B_V$  ‘scalar’ in order to highlight this difference, but emphasise that for the remainder ‘second variation’ refers exclusively to the quadratic form  $B_V$  from Definition 2.7. (For the same reasons we also call the Jacobi operator  $L_V$  and its spectrum ‘scalar’ in Definitions 2.6 and 2.7 below, but drop this adjective in the remainder of the text.)

After integrating by parts on  $\text{reg } V$ , the form  $B_V$  corresponds to the second-order elliptic operator  $L_V = \Delta_V + |A|^2 + \text{Ric}_M(\nu, \nu)$ , where  $\Delta_V$  is the Laplacian on  $\text{reg } V$ .

**Definition 2.6** (Scalar Jacobi operator). The *scalar Jacobi operator* of  $V$ , denoted  $L_V$ , is the second-order elliptic operator

$$(2.8) \quad L_V \phi = \Delta_V \phi + (|A|^2 + \text{Ric}_M(\nu, \nu))\phi \quad \text{for all } \phi \in C^2(\text{reg } V),$$

where  $\nu : \text{reg } V \rightarrow NV$  is an arbitrary measurable unit normal vector field.

The curvature of  $\text{reg } V$  can blow up as one approaches  $\text{sing } V$ , in which case the coefficients of the operator  $L_V$  would not be bounded. To avoid this, we restrict ourselves to an open subset  $W \subset \subset U \setminus \text{sing } V$ ; moreover we require  $W \cap \text{reg } V \neq \emptyset$  to avoid vacuous statements. By standard elliptic PDE theory [GT98, Ch. 8] the spectrum  $\lambda_1 \leq \lambda_2 \leq \dots \rightarrow +\infty$  of  $L_V$  in  $W$  is discrete and bounded below, and the eigenvectors of  $L_V$  span the space  $H_0^1(W \cap \text{reg } V) = W_0^{1,2}(W \cap \text{reg } V)$ , which we will abbreviate throughout by  $H_0^1$ . (We use the sign convention defined in [GT98, Ch. 8], where  $\lambda \in \mathbb{R}$  is an eigenvalue of  $L_V$  if there is  $\varphi \in H_0^1$  such that  $L_V \varphi + \lambda \varphi = 0$ .)

**Definition 2.7** (Scalar stability spectrum). The sequence  $(\lambda_p)_{p \in \mathbb{N}}$  is called the *scalar stability spectrum* of  $V$  in  $W$ . We write  $(\lambda_p(W))_{p \in \mathbb{N}}$  when we want to emphasise the subset  $W \subset \subset U \setminus \text{sing } V$ .

**Definition 2.8** (Index of  $B_V$ ). The *index* of  $B_V$  in  $W$  is the maximal dimension of a subspace of  $H_0^1$  on which  $B_V$  is negative definite; equivalently

$$(2.9) \quad \text{ind}_W B_V = \text{card}\{p \in \mathbb{N} \mid \lambda_p(W) < 0\}.$$

Moreover  $\text{ind } B_V := \sup_W (\text{ind}_W B_V)$ , where the supremum is taken over all  $W \subset \subset U \setminus \text{sing } V$  with  $W \cap \text{reg } V \neq \emptyset$ .

*Remark 2.9.* The index of  $B_V$  coincides with the Morse index of  $\text{reg } V$  with respect to the area functional when  $\text{reg } V$  is two-sided.

**2.3. Statement of main theorem.** Let  $(\epsilon_i)$  be a sequence of positive parameters with  $\epsilon_i \rightarrow 0$  and consider an associated sequence of functions  $(u_i)$  in  $C^3(U)$  satisfying the following hypotheses:

(A) Every  $u_i \in C^3(U)$  is a critical point of the Allen–Cahn functional

$$(2.10) \quad E_{\epsilon_i}(u) = \int_U \epsilon_i \frac{|\nabla u|^2}{2} + \frac{W(u)}{\epsilon_i} d\mathcal{H}^{n+1},$$

i.e.  $u_i$  satisfies the equation

$$(2.11) \quad -\epsilon_i^2 \Delta u_i + W'(u_i) = 0 \quad \text{in } U.$$

(B) There exist constants  $C, E_0 < \infty$  such that

$$(2.12) \quad \sup_i \|u_i\|_{L^\infty(U)} \leq C \quad \text{and} \quad \sup_i E_{\epsilon_i}(u_i) \leq E_0.$$

(C) There exists an integer  $k \geq 0$  such that the Morse index of each  $u_i$  is at most  $k$ , i.e. any subspace of  $C_c^1(U)$  on which the second variation

$$(2.13) \quad \delta^2 E_{\epsilon_i}(u_i)(\phi, \phi) = \int_U \epsilon_i |\nabla \phi|^2 + \frac{W''(u_i)}{\epsilon_i} \phi^2 d\mathcal{H}^{n+1}$$

is negative definite has dimension at most  $k$ . We write this  $\text{ind } u_i \leq k$ , and if  $k = 0$ , say that  $u_i$  is *stable in  $U$* .

*Remark 2.10.* Generally for open  $U' \subset U$ ,  $\text{ind}_{U'} u_i$  denotes the index of  $\delta^2 E_{\epsilon_i}(u_i)$  with respect to variations in  $H_0^1(U')$ . When  $\text{ind}_{U'} u_i = 0$ , we say that  $u_i$  is *stable in  $U'$* .

We follow Tonegawa [Ton05], using an idea originally developed by Ilmanen [Ilm93] in a parabolic setting, and ‘average the level sets’ of  $u_i \in C^3(U)$  by defining a varifold  $V^i$ .

**Definition 2.11.** Define the varifold  $V^i$  by

$$(2.14) \quad V^i(\phi) = \frac{1}{\sigma} \int_{U \cap \{\nabla u_i \neq 0\}} \epsilon_i \frac{|\nabla u_i(x)|^2}{2} \phi(x, T_x\{u_i = u_i(x)\}) d\mathcal{H}^{n+1}(x)$$

for all  $\phi \in C_c(G_n(U))$ . Here  $T_x\{u_i = u_i(x)\}$  is the tangent space to the level set  $\{u_i = u_i(x)\}$  at  $x \in U$ , and  $\sigma = \int_{-1}^1 \sqrt{W(s)/2} ds$  is a constant.

*Remark 2.12.* In [HT00, Gua15]  $V^i$  is defined by the slightly different expression  $V^i(\phi) = \frac{1}{\sigma} \int_{U \cap \{\nabla u_i \neq 0\}} |\nabla w_i(x)| \phi(x, T_x\{u_i = u_i(x)\}) d\mathcal{H}^{n+1}(x)$ , with  $w_i$  as in Theorem 2.13. The ‘equipartition of energy’ (2.16) from Theorem 2.13 shows that the two definitions give rise to the same limit varifold  $V$ .

The weight measures  $\|V^i\|$  of these varifolds satisfy

$$(2.15) \quad \|V^i\|(A) = \frac{1}{\sigma} \int_{A \cap \{\nabla u_i \neq 0\}} \epsilon_i \frac{|\nabla u_i|^2}{2} d\mathcal{H}^{n+1} \leq \frac{E_0}{\sigma}$$

for all Borel subsets  $A \subset U$ , where the inequality follows from the energy bound in Hypothesis (B). The resulting bound  $V^i(G_n(U)) \leq \frac{E_0}{\sigma}$  allows us to extract a subsequence that converges to a varifold  $V$ , with properties laid out in the following theorem by Hutchinson–Tonegawa [HT00].

**Theorem 2.13** ([HT00]). *Let  $(u_i)$  be a sequence in  $C^3(U)$  satisfying Hypotheses (A) and (B). Passing to a subsequence  $V^i \rightharpoonup V$  as varifolds, and*

- (a)  *$V$  is a stationary integral varifold,*
- (b)  $\|V\|(U) = \liminf_{i \rightarrow \infty} \frac{1}{2\sigma} E_{\epsilon_i}(u_i),$
- (c) *for all  $\phi \in C_c(U)$ :*

$$(2.16) \quad \lim_{i \rightarrow \infty} \int_U \epsilon_i \frac{|\nabla u_i|^2}{2} \phi = \lim_{i \rightarrow \infty} \int_U \frac{W(u_i)}{\epsilon_i} \phi = \lim_{i \rightarrow \infty} \int_U |\nabla w_i| \phi,$$

where  $w_i := \Psi \circ u_i$  and  $\Psi(t) := \int_0^t \sqrt{W(s)/2} ds$ .

Up to a factor of  $\epsilon_i$  the second variation  $\delta^2 E_{\epsilon_i}$  corresponds to the second-order elliptic operator  $L_i := \Delta - \epsilon_i^{-2} W''(u_i)$ . As in the discussion for the Jacobi operator,  $L_i$  has discrete spectrum  $\lambda_1^i \leq \lambda_2^i \leq \dots \rightarrow +\infty$ , which we denote by  $(\lambda_p^i(W))_{p \in \mathbb{N}}$  when we want to emphasise its dependence on  $W$ . Our main result is the following theorem.

**Theorem A.** *Let  $M^{n+1}$  be a closed Riemannian manifold, and  $U \subset M$  an open subset. Let  $(u_i)$  be a sequence in  $C^3(U)$  satisfying Hypotheses (A), (B) and (C), and  $V^i \rightharpoonup V$ . Then  $\dim_{\mathcal{H}} \text{sing } V \leq n - 7$  and*

- (i)  $\lambda_p(W) \geq \limsup_{i \rightarrow \infty} \lambda_p^i(W)$  for all open  $W \subset\subset U \setminus \text{sing } V$  with  $W \cap \text{reg } V \neq \emptyset$ ,
- (ii)  $\text{ind } B_V \leq k$ .

*Remark 2.14.* The spectral lower bound remains true if the assumptions are weakened and one assumes that for some  $k \in \mathbb{N}$  there is  $\mu \in \mathbb{R}$  such that  $\lambda_k^i \geq \mu$  for all  $i$  instead of an index bound—this observation is inspired by [ACS15], where a similar generalisation is made in the context of minimal surfaces. One obtains the spectral bound via an inductive argument on  $k$  similar to the argument in Section 4, noting for the base case of the induction that bounds as in Corollary 3.5 hold if  $\lambda_1^i \geq \mu$ .

The following corollary is an immediate consequence of Theorem A.

**Corollary B.** *If  $\text{reg } V$  is two-sided, then its Morse index with respect to the area functional satisfies  $\text{ind}_{\mathcal{H}^n} \text{reg } V \leq k$ .*

If the limit varifold arising from Guaraco’s 1-parameter min-max construction [Gua15] has two-sided  $\text{reg } V$ , then by Corollary B its Morse index is at most 1.

### 3. PRELIMINARY RESULTS

The preliminary results are divided into three parts. In the first, following [Ton05] we introduce ‘second fundamental forms’  $A^i$  for the varifolds and relate them to the second variation of the Allen–Cahn functional. The last two sections are dedicated to the spectra of the operators  $L_V = \Delta_V + |A|^2 + \text{Ric}_M(\nu, \nu)$  and  $L_i = \Delta - \epsilon_i^{-2} W''(u_i)$ .

**3.1. Stability and  $L^2$ –bounds of curvature.** To simplify the discussion fix for the moment a critical point  $u \in C^3(U) \cap L^\infty(U)$  of the Allen–Cahn functional  $E_\epsilon$ , with associated varifold  $V^\epsilon$  defined by (2.11).

Let  $x \in U$  be a regular point of  $u$ , that is  $\nabla u(x) \neq 0$ . In a small enough neighbourhood of  $x$ , the level set  $\{u = u(x)\}$  is embedded in  $M$ . Call  $\Sigma \subset M$  this embedded portion of the level set, and let  $A^\Sigma$  be its second fundamental form.

**Definition 3.1** (Second fundamental form of  $V^\epsilon$ ). The function  $A^\epsilon$  is defined at all  $x \in U$  where  $\nabla u(x) \neq 0$  by  $A^\epsilon(x) = A^\Sigma(x) \in T_x^* M^{\otimes 2} \otimes T_x M$ .

*Remark 3.2.* Second fundamental forms can be generalised to the context of varifolds via the integral identity (B.21)—see Appendix B, or [Hut86] for the original account. Strictly speaking it is an abuse of language to call  $A^\epsilon$  the ‘second fundamental form’ of  $V^\epsilon$ , as it satisfies this identity only up to a small error term (B.15).

By definition  $\nabla_X Y = \nabla_X^\Sigma Y + A^\Sigma(X, Y)$  for all  $X, Y \in C^1(T\Sigma)$ . Making implicit use of the musical isomorphisms here and throughout the text, write  $\nu^\epsilon(x) = \frac{\nabla u(x)}{|\nabla u(x)|}$ , so that

$$(3.1) \quad A^\Sigma(X, Y) = \langle \nabla_X Y, \nu^\epsilon \rangle \nu^\epsilon = -\langle Y, \nabla_X \nu^\epsilon \rangle \nu^\epsilon.$$



**Lemma 3.3.** *Let  $x \in U$  be a regular point of  $u$ . Then*

$$(3.2) \quad |A^\epsilon|(x)^2 \leq \frac{1}{|\nabla u|^2(x)} (|\nabla^2 u|^2(x) - |\nabla|\nabla u||^2(x)),$$

where  $\nabla^2 u(x)$  is the Hessian of  $u$  at  $x$ .

*Proof.* The second fundamental form  $A^\Sigma$  is expressed in terms of the covariant derivative  $\nabla\nu^\epsilon$  by

$$(3.3) \quad A^\Sigma = -\nabla\nu^\epsilon|_{T\Sigma \otimes T\Sigma} \otimes \nu^\epsilon.$$

We can express  $\nabla\nu^\epsilon$  as

$$(3.4) \quad \nabla\nu^\epsilon = \frac{\nabla^2 u}{|\nabla u|} - \nu^\epsilon \otimes \frac{\nabla|\nabla u|}{|\nabla u|},$$

whence after restriction to  $T\Sigma \otimes T\Sigma$  we get

$$(3.5) \quad A^\Sigma = -\frac{1}{|\nabla u|} \nabla^2 u|_{T\Sigma \otimes T\Sigma} \otimes \nu^\epsilon.$$

On the other hand  $\nabla|\nabla u| = \langle \nabla^2 u, \nu^\epsilon \rangle$  wherever  $\nabla u \neq 0$ , so after decomposing the Hessian  $\nabla^2 u$  in terms of its action on  $T\Sigma$  and  $N\Sigma$ , we obtain

$$(3.6) \quad |\nabla^2 u|^2 - |\nabla|\nabla u||^2 = |\nabla u|^2 |A^\Sigma|^2 + |\nabla^2 u|_{T\Sigma \otimes N\Sigma}|^2 \geq |\nabla u|^2 |A^\epsilon|^2. \quad \square$$

When considering the second variation, it somewhat simplifies notation to rescale the energy as  $\mathcal{E}_\epsilon = \epsilon^{-1} E_\epsilon$ . Its second variation is  $\delta^2 \mathcal{E}_\epsilon(u)(\phi, \phi) = \int_U |\nabla \phi|^2 + \frac{W''(u)}{\epsilon^2} \phi^2$ , defined for all  $\phi \in C_c^1(U)$ , which by a density argument can be extended to  $H_0^1(U)$ . The following identity will be useful throughout; a proof can be found in either of the indicated sources.

**Lemma 3.4** ([FSV13, Ton05]). *Let  $u \in C^3(U) \cap L^\infty(U)$  be a critical point of  $E_\epsilon$ . For all  $\phi \in H_0^1(U)$ :*

$$(3.7) \quad \delta^2 \mathcal{E}_\epsilon(u)(|\nabla u| \phi, |\nabla u| \phi) = \int_U |\nabla u|^2 |\nabla \phi|^2 - \int_{U \cap \{\nabla u \neq 0\}} (\text{Ric}_M(\nabla u, \nabla u) + |\nabla^2 u|^2 - |\nabla|\nabla u||^2) \phi^2.$$

Combining (3.7) with the  $\|V^\epsilon\|$ -a.e. bound (3.2) yields for all  $\phi \in H_0^1(U)$

$$(3.8) \quad \frac{\epsilon}{2} \delta^2 \mathcal{E}_\epsilon(u)(|\nabla u| \phi, |\nabla u| \phi) \leq \int |\nabla \phi|^2 - (|A^\epsilon|^2 + \text{Ric}(\nu^\epsilon, \nu^\epsilon)) \phi^2 d\|V^\epsilon\|.$$

When  $u$  is stable, that is when  $\delta^2 E_\epsilon(u)$  is non-negative, then this identity yields  $L^2(V^\epsilon)$ -bounds for  $A^\epsilon$ .

**Corollary 3.5.** *There is a constant  $C = C(M) > 0$  such that if  $u \in C^3(U) \cap L^\infty(U)$  is a critical point of  $E_\epsilon$  and is stable in the open ball  $B(x, r) \subset U$  of radius  $r \leq 1$  then*

$$(3.9) \quad \int_{B(x, \frac{r}{2})} |A^\epsilon|^2 d\|V^\epsilon\| \leq \frac{C}{r^2} \|V^\epsilon\|(B(x, r)).$$

*Proof.* The Ricci curvature term in (3.8) can be bounded by some constant  $C(M) \geq 1$  as the ambient manifold  $M$  is closed, so  $\int_{B(x,r)} |A^\epsilon|^2 \phi^2 d\|V^\epsilon\| \leq C(M) \int \phi^2 + |\nabla \phi|^2 d\|V^\epsilon\|$  for all  $\phi \in C_c^1(B(x,r))$ . Plug in a cut-off function  $\eta \in C_c^1(B(x,r))$  with  $\eta = 1$  in  $B(x, \frac{r}{2})$  and  $|\nabla \eta| \leq 3r^{-1}$  to obtain the desired inequality.  $\square$

Now we turn to a sequence  $(u_i)$  of critical points satisfying Hypotheses (A)–(C). If the  $u_i$  are stable in a ball as described in Corollary 3.5, then the uniform weight bounds (2.15) imply uniform  $L^2(V^i)$ –bounds on the second fundamental forms  $A^i := A^{\epsilon_i}$ . Under these conditions the  $A^i$  converge weakly to the second fundamental form of  $V$ .

**Proposition 3.6.** *Let  $W \subset\subset U \setminus \text{sing } V$  be open with  $W \cap \text{reg } V \neq \emptyset$ . If  $\sup_i \int_W |A^i|^2 d\|V^i\| < +\infty$ , then some subsequence  $A^{i'} dV^{i'} \rightharpoonup A dV$  weakly as Radon measures on  $G_n(W)$ , and*

$$(3.10) \quad \int_W |A|^2 d\|V\| \leq \liminf_{i \rightarrow \infty} \int_W |A^i|^2 d\|V^i\|,$$

where  $A$  is the second fundamental form of  $\text{reg } V \subset M$ .

The weak subsequential convergence follows immediately from compactness of Radon measures; the main difficulty is to show that the weak limit is  $A dV$ . The proof is a straight-forward adaptation of the argument used for the stable case in [Ton05]; we present a complete argument in Appendix B for the reader's convenience.

**3.2. Spectrum of  $L_V$  and weighted min-max.** Throughout we restrict ourselves to an open subset  $W \subset\subset U \setminus \text{sing } V$  to avoid blow-up of the coefficients of  $L_V$  near the singular set, and assume  $W \cap \text{reg } V \neq \emptyset$  to avoid vacuous statements. As  $W \cap \text{reg } V$  is compactly contained in  $\text{reg } V$ , it can only intersect finitely many connected components  $C_1, \dots, C_N$  of  $\text{reg } V$ . By the constancy theorem [Sim84, Thm. 41.1] the multiplicity function  $\theta$  of a stationary integral varifold  $V$  is constant on every connected component of  $\text{reg } V$ ; we write  $\theta_1, \dots, \theta_N$  for the respective multiplicities of  $C_1, \dots, C_N$ .

By the classical theory of elliptic PDE [GT98, Ch. 8], the spectrum of  $L_V$  has the following min-max characterisation:

$$(3.11) \quad \lambda_p = \inf_{\dim S=p} \max_{\phi \in S \setminus 0} \frac{B_{\text{reg } V}(\phi, \phi)}{\|\phi\|_{L^2}^2} \quad \text{for all } p \in \mathbb{N},$$

where  $S$  is a linear subspace of  $H_0^1$  (which we recall is our abbreviated notation for  $H_0^1(W \cap \text{reg } V)$ ). From this we easily obtain a min-max characterisation that is ‘weighted’ by the multiplicities  $\theta_1, \dots, \theta_N$ :

$$(3.12) \quad \lambda_p = \inf_{\dim S=p} \max_{\phi \in S \setminus 0} \frac{B_V(\phi, \phi)}{\|\phi\|_{L^2(V)}^2} \quad \text{for all } p \in \mathbb{N}.$$

To see this, make the following elementary observation: as functions  $\phi \in H_0^1$  vanish near the boundary of every connected component  $C \subset \text{reg } V$ , the function  $\phi_C$  defined on  $W \cap \text{reg } V$  by

$$(3.13) \quad \phi_C = \begin{cases} \phi & \text{on } C \\ 0 & \text{on } W \cap \text{reg } V \setminus C \end{cases}$$

also belongs to  $H_0^1$ . Moreover

$$(3.14) \quad B_V(\phi_C, \phi_C) = \theta_C B_{\text{reg } V}(\phi_C, \phi_C) \text{ and } \|\phi_C\|_{L^2(V)}^2 = \theta_C \|\phi_C\|^2,$$

where  $\theta_C$  denotes the multiplicity of  $C$ . Define a linear isomorphism of  $H_0^1$  by mapping  $\phi \mapsto \bar{\phi} := \sum_{j=1}^N \theta_j^{-1/2} \phi_{C_j}$ ; then

$$(3.15) \quad \frac{B_V(\bar{\phi}, \bar{\phi})}{\|\bar{\phi}\|_{L^2}^2} = \frac{B_{\text{reg } V}(\phi, \phi)}{\|\phi\|_{L^2(V)}^2}.$$

Therefore the ‘unweighted’ and ‘weighted’ min-max characterisations (3.11) and (3.12) are in fact equivalent. In the remainder we mainly use (3.12), and abbreviate this as  $\lambda_p = \inf_{\dim S=p} \max_{S \setminus \{0\}} J_V$ , where  $J_V$  denotes the ‘weighted’ Rayleigh quotient

$$(3.16) \quad J_V(\phi) = \frac{B_V(\phi, \phi)}{\|\phi\|_{L^2(V)}^2} \quad \text{for all } \phi \in H_0^1 \setminus \{0\}.$$

The min-max characterisation implies the following lemma, which highlights the dependence of the spectrum  $\lambda_p(W)$  on the subset  $W$ .

**Lemma 3.7.**

- (a) *If  $W_1 \subset W_2 \subset\subset U \setminus \text{sing } V$ , then  $\lambda_p(W_1) \geq \lambda_p(W_2)$ : the spectrum is monotone decreasing.*
- (b) *If  $W_1, W_2 \subset\subset U \setminus \text{sing } V$  have  $W_1 \cap W_2 = \emptyset$ , then  $\text{ind}_{W_1} B_V + \text{ind}_{W_2} B_V = \text{ind}_{W_1 \cup W_2} B_V$ .*
- (c) *If  $W \subset\subset U \setminus \text{sing } V$  and  $y \in W \cap \text{reg } V$ , then  $\lambda_p(W) = \lim_{R \rightarrow 0} \lambda_p(W \setminus \overline{B}(y, R))$ .*

*Remark 3.8.* The same properties hold for the spectrum and index of  $L_i$ , and the proof is easily modified to cover this case.

*Proof.* (a) This is immediate from the min-max characterisations, or simply by definition of the spectrum. Similarly for (b).

(c) By monotonicity of the spectrum we have for all  $R > R' > 0$ :

$$(3.17) \quad \lambda_p(W \setminus \overline{B}(y, R)) \geq \lambda_p(W \setminus \overline{B}(y, R')) \geq \lambda_p(W),$$

so the limit as  $R \rightarrow 0$  exists and is bounded below by  $\lambda_p(W)$ . It remains only to show that  $\lim_{R \rightarrow 0} \lambda_p(W \setminus \overline{B}(y, R)) \leq \lambda_p(W)$ .

By monotonicity of the spectrum it is equivalent to show that for every fixed radius  $R > 0$ ,  $\lim_{m \rightarrow \infty} \lambda_p(W \setminus \overline{B}(y, 2^{-m}R)) \leq \lambda_p(W)$ . Let  $(\rho_m)_{m \in \mathbb{N}}$  be a sequence in  $C_c^1(B(y, R) \cap \text{reg } V)$  with the following properties:

- (1)  $\rho_m|_{B(y, 2^{-m}R) \cap \text{reg } V} \equiv 0$  and  $\rho_m \rightarrow 1$   $\mathcal{H}^n$ -a.e. in  $W \setminus \{y\} \cap \text{reg } V$ ,
- (2)  $\|\nabla_V \rho_m\|_{L^2(W \cap \text{reg } V)} \rightarrow 0$ .

Let a small  $\delta > 0$  be given and choose a family  $(\phi_1, \dots, \phi_p)$  in  $C_c^1(W \cap \text{reg } V)$  with  $\text{span}(\phi_1, \dots, \phi_p) =: S$  satisfying  $\max_{S \setminus \{0\}} J_V \leq \lambda_p(W) + \delta$ . Write  $\rho_m S$  for the  $p$ -plane  $\text{span}(\rho_m \phi_1, \dots, \rho_m \phi_p) \subset C_c^1(W \setminus \overline{B}(y, 2^{-m}R) \cap \text{reg } V)$ . By the weighted min-max formula (3.12):

$$(3.18) \quad \max_{\rho_m S \setminus \{0\}} J_V \geq \lambda_p(W \setminus \overline{B}(y, 2^{-m}R)).$$

Let  $t_m \in \mathbb{S}^{p-1} \subset \mathbb{R}^p$  denote the coefficients of the linear combination  $t_m \cdot \rho_m \phi := \rho_m \sum_{j=1}^p t_{mj} \phi_j \in \rho_m S$  that realises  $\max_{\rho_m S \setminus \{0\}} J_V$ . Passing to a

convergent subsequence  $t_m \rightarrow t \in \mathbb{S}^{p-1} \subset \mathbb{R}^p$  we get  $J_V(t_m \cdot \rho_m \phi) \rightarrow J_V(t \cdot \phi)$ , and hence

$$(3.19) \quad \lim_{m \rightarrow \infty} J_V(t_m \cdot \rho_m \phi) \leq \max_{S \setminus \{0\}} J_V.$$

On the one hand  $\max_{S \setminus \{0\}} J_V \leq \lambda_p(W) + \delta$  by our choice of  $S$ . On the other hand  $\lim_{m \rightarrow \infty} \lambda_p(W \setminus \overline{B}(y, 2^{-m}R)) \leq \lim_{m \rightarrow \infty} J_V(t_m \cdot \rho_m \phi)$  by our choice of  $t_m$ . The conclusion follows after combining these two observations and letting  $\delta \rightarrow 0$ .  $\square$

*Remark 3.9.* A sequence of functions  $(\rho_m)$  with properties (1) and (2) exists provided  $n \geq 2$ , as we assume throughout. When  $n \geq 3$  one can use the standard cutoff functions; for  $n = 2$  a more precise construction is necessary, described for instance in [EG15, Sec. 4.7].

**3.3. Spectrum of  $L_i$  and conditional proof of Theorem A.** The main result in this section is Lemma 3.13; essentially it says that

$$(3.20) \quad \lambda_p(W) \geq \limsup_{i \rightarrow \infty} \lambda_p^i(W)$$

holds under the condition that  $\sup_i \int_W |A^i|^2 d\|V^i\| < +\infty$ . What precedes it in this section are technical results required for its proof.

Classical elliptic PDE theory [GT98, Ch. 8] explains that the eigenvalues  $\lambda_p^i(W)$  of  $L_i$  on  $H_0^1(W)$  have the following min-max characterisation in terms of the rescaled Allen–Cahn functional  $\mathcal{E}_{\epsilon_i} = \epsilon_i^{-1} E_{\epsilon_i}$ :

$$(3.21) \quad \lambda_p^i(W) = \inf_{\dim S=p} \max_{\phi \in S \setminus \{0\}} \frac{\delta^2 \mathcal{E}_{\epsilon_i}(u_i)(\phi, \phi)}{\|\phi\|_{L^2}^2} \quad \text{for all } p \in \mathbb{N},$$

where  $S \subset H_0^1(W)$  is a linear subspace. Define the *Rayleigh quotient*  $J_i$  by

$$(3.22) \quad J_i(\phi) = \frac{\delta^2 \mathcal{E}_{\epsilon_i}(u_i)(\phi, \phi)}{\|\phi\|_{L^2}^2} \quad \text{for all } \phi \in H_0^1(W) \setminus \{0\},$$

so that we can write this more succinctly as  $\lambda_p^i = \inf_{\dim S=p} \max_{S \setminus \{0\}} J_i$ . To compare the spectra of  $L_i$  on  $H_0^1(W)$  with the spectrum of  $L_V$  on  $H_0^1 = H_0^1(W \cap \text{reg } V)$ , we extend functions in  $C_c^1(W \cap \text{reg } V)$  to  $C_c^1(W)$ .

Pick a small enough  $0 < \tau < \text{inj}(M)$  so that  $B_\tau V := \exp N_\tau V$  is a tubular neighbourhood of  $W \cap \text{reg } V$ , where  $N_\tau V := \{s_p \in NV \mid p \in W \cap \text{reg } V, |s_p| < \tau\}$ . We abuse notation slightly to denote points in  $B_\tau V$  by  $s_p$ , and identify the fibre  $N_p V$  with  $(\exp_p)_* N_p V \subset T_{s_p}(B_\tau V)$ . The distance function  $d_V : x \in B_\tau V \mapsto \text{dist}(x, \text{reg } V)$  is Lipschitz and smooth on  $B_\tau V \setminus \text{reg } V$ . By the Gauss lemma  $\text{grad } d_V(s_p) = -s_p/|s_p|$  for all  $s_p \in B_\tau V \setminus \text{reg } V$ . A function  $\phi \in C^1(B_\tau V)$  is constant along geodesics normal to  $\text{reg } V$  if  $\phi(s_p) = \phi(0_p)$  for all  $s_p \in B_\tau V$ , or equivalently if and only if  $\langle \nabla \phi, \nabla d_V \rangle \equiv 0$  in  $B_\tau V \setminus \text{reg } V$ .

**Lemma 3.10.** *Any  $\phi \in C_c^1(W \cap \text{reg } V)$  can be extended to  $C_c^1(W)$  with  $\langle \nabla \phi, \nabla d_V \rangle \equiv 0$  in  $B_{\frac{\tau}{2}} V \setminus \text{reg } V$  for some  $\tau = \tau(\phi) > 0$ .*

*Proof.* Extend  $\phi \in C_c^1(W \cap \text{reg } V)$  to  $B_\tau V$  by setting  $\tilde{\phi}(s_p) = \phi(p)$ , so that  $\langle \nabla d_V, \nabla \tilde{\phi} \rangle \equiv 0$  in  $B_\tau V \setminus \text{reg } V$ . Let  $\eta \in C^1[0, \infty)$  be a cutoff function with  $0 \leq \eta \leq 1$ ,  $\eta \equiv 1$  on  $[0, 1/2)$  and  $\text{spt } \eta \subset [0, 1)$ . Then

$$(3.23) \quad (\eta \circ d_V / \tau) \tilde{\phi} \in C_c^1(B_\tau V) \text{ and } (\eta \circ d_V / \tau) \tilde{\phi} = \tilde{\phi} \text{ on } B_{\tau/2} V.$$

Moreover even though  $B_\tau V \not\subset W$  in general, as  $\text{spt } \phi$  is compactly contained in  $W \cap \text{reg } V$  we still have  $(\eta \circ d_V/\tau)\tilde{\phi} \in C_c^1(W)$  provided  $0 < \tau < \text{dist}(\text{spt } \phi, \partial W)$ .  $\square$

The following lemma gives an asymptotic lower bound for the Rayleigh quotient  $J_V$  in terms of the  $J_i$ .

**Lemma 3.11.** *Let  $B_\tau V$  be a tubular neighbourhood of  $W \cap \text{reg } V$  with width  $\tau > 0$ , and let  $(\phi_i)_{i \in \mathbb{N}}$  be a sequence of functions in  $C_c^1(W)$  with*

- (a)  $\langle \nabla \phi_i, \nabla d \rangle \equiv 0$  in  $W \cap B_{\tau/2} V$  for all  $i$ ,
- (b)  $\phi_i \rightarrow \phi$  in  $C_c^1(W)$  as  $i \rightarrow \infty$ , where  $\phi \neq 0$  in  $W \cap \text{reg } V$ .

*If  $\sup_i \int_W |A^i|^2 d\|V^i\| < +\infty$ , then  $J_V(\phi) \geq \limsup_{i \rightarrow \infty} J_i(|\nabla u_i| \phi_i)$ .*

*Proof.* Before we start the proof proper, note that for all  $\phi \in H_0^1(U)$ , dividing both sides of (3.8) by  $\frac{\epsilon_i}{2} \int \phi^2 |\nabla u_i|^2 d\mathcal{H}^{n+1} = \int \phi^2 d\|V^i\|$  yields

$$(3.24) \quad \|\phi\|_{L^2(V^i)}^{-2} \int |\nabla \phi|^2 - (|A^i|^2 + \text{Ric}(\nu_i, \nu_i)) \phi^2 d\|V^i\| \geq J_i(|\nabla u_i| \phi),$$

provided of course that  $\|\phi\|_{L^2(V^i)}^2 \neq 0$ .

We prove the lower bound  $J_V(\phi) \geq \limsup_{i \rightarrow \infty} J_i(|\nabla u_i| \phi)$  by reasoning with the expression on the left-hand side instead—we do this by treating the terms in the expression separately in the calculations (1)–(3). Combining (1) with our assumption that  $\|\phi\|_{L^2(V)}^2 \neq 0$ , we obtain that  $\|\phi_i\|_{L^2(V^i)} = 0$  for at most finitely many  $i$ . The Rayleigh quotient  $J_i(|\nabla u_i| \phi_i)$  is thus well-defined for large enough  $i$ , and the conclusion follows by combining (3.24) with

- (1)  $\int \phi^2 d\|V\| = \lim_{i \rightarrow \infty} \int \phi_i^2 d\|V^i\|$
- (2)  $\int |\nabla_V \phi|^2 d\|V\| = \lim_{i \rightarrow \infty} \int |\nabla \phi_i|^2 d\|V^i\|$ ,
- (3)  $\int |A|^2 \phi^2 d\|V\| \leq \liminf_{i \rightarrow \infty} \int |A^i|^2 \phi_i^2 d\|V^i\|$ ,
- (4)  $\int \text{Ric}_M(\nu, \nu) \phi^2 d\|V\| = \lim_{i \rightarrow \infty} \int \text{Ric}_M(\nu_i, \nu_i) \phi_i^2 d\|V^i\|$ .

(1) By assumption  $\phi_i^2 \rightarrow \phi^2$  in  $C_c(W)$ , whence by Corollary A.3 we get  $\int \phi^2 d\|V\| = \lim_{i \rightarrow \infty} \int \phi_i^2 d\|V^i\|$ . (2) follows similarly, after noticing that  $\langle \nabla \phi_i, \nabla d_V \rangle \equiv 0$  implies  $|\nabla \phi|^2 = |\nabla_V \phi|^2$  on  $W \cap \text{reg } V$ .

(3) The sequence  $(A^i \phi_i d\|V^i\|)$  converges weakly to  $A \phi d\|V\|$ , as we can show by testing against an arbitrary  $\varphi \in C_c(U)$ :

$$(3.25) \quad \int A^i \phi_i \varphi d\|V^i\| - \int A \phi \varphi d\|V\| = \int A^i (\phi_i - \phi) \varphi d\|V^i\| + \int A^i \phi \varphi d\|V^i\| - \int A \phi \varphi d\|V\|.$$

The first integral  $|\int A^i (\phi_i - \phi) \varphi d\|V^i\|| \leq \|\phi_i - \phi\|_{L^\infty} \|\varphi\|_{L^2(V^i)} \|A^i\|_{L^2(V^i)} \rightarrow 0$  as  $\phi_i \rightarrow \phi$  in  $C_c(W)$  as  $i \rightarrow \infty$ . The remaining terms tend to 0 by the weak convergence of  $A^i d\|V^i\| \rightharpoonup A d\|V\|$  tested against  $\phi \varphi \in C_c(W)$ . Inequality (A.4) from Lemma A.1 gives  $\int |A|^2 \phi^2 d\|V\| \leq \liminf_{i \rightarrow \infty} \int |A^i|^2 \phi_i^2 d\|V^i\|$ .

(4) For each  $S \in G_n(T_p M)$  pick a unit vector  $\nu_S \in T_p M$  normal to  $S$ , and define a smooth function  $R_M$  on  $G_n(U)$  by  $R_M : S \mapsto \text{Ric}_M(\nu_S, \nu_S)$ . Then

$\phi_i^2 R_M \rightarrow \phi^2 R_M$  in  $C_c(G_n(U))$  as  $i \rightarrow \infty$ , and by Corollary A.3:

$$(3.26) \quad \int \phi_i^2 \operatorname{Ric}_M(\nu_i, \nu_i) d\|V^i\| = \int \phi_i^2 R_M dV^i \rightarrow \int \phi^2 R_M dV = \int \phi^2 \operatorname{Ric}_M(\nu, \nu) d\|V\|. \quad \square$$

In the proof of Lemma 3.13 one needs to make sure that a linearly independent family in  $C_c^1(W)$  remains independent when multiplied by  $|\nabla u_i|$ . This follows from Lemma 3.12 provided—as we may later assume without restricting generality—that there is no connected component  $C \subset W$  with  $C \cap \operatorname{reg} V = \emptyset$ .

**Lemma 3.12.**

- (a) *For all  $i$ ,  $U \cap \operatorname{spt}\|V^i\|$  is open in  $U$ . (As it also closed, it is equal to a union of connected components of  $U$ .)*
- (b) *If every connected component of  $W$  intersects  $\operatorname{spt}\|V\|$  then there is  $I(W) \in \mathbb{N}$  such that for all  $i \geq I(W)$ ,  $W \subset \operatorname{spt}\|V^i\|$ .*
- (c) *If every connected component of  $W$  intersects  $\operatorname{spt}\|V\|$  and  $(\phi_1, \dots, \phi_p)$  is linearly independent in  $H_0^1(W)$ , then so is  $(|\nabla u_i|\phi_1, \dots, |\nabla u_i|\phi_p)$  for all  $i \geq I(W)$ .*

*Proof.* (a) Recall that  $\|V^i\| = \epsilon_i \frac{|\nabla u_i|^2}{2} d\mathcal{H}^{n+1}$  and note that

$$(3.27) \quad U = \operatorname{cl}\{\nabla u_i \neq 0\} \cup \operatorname{int}\{\nabla u_i = 0\},$$

$$(3.28) \quad U \cap \operatorname{spt}\|V^i\| = \operatorname{cl}\{\nabla u_i \neq 0\},$$

where  $\operatorname{cl}$  and  $\operatorname{int}$  denote the topological closure and interior respectively. To conclude, it suffices to show that a connected component  $C \subset U$  cannot simultaneously intersect  $\operatorname{int}\{\nabla u_i = 0\}$  and  $\operatorname{cl}\{\nabla u_i \neq 0\}$ . Indeed, differentiating the  $\epsilon_i$ -Allen–Cahn equation (2.11) shows that the 1-form  $\alpha := \nabla u_i \in \Omega^1(U)$  satisfies the equation

$$(3.29) \quad \Delta \alpha - \epsilon_i^{-2} W''(u_i) \alpha = 0 \text{ in } U,$$

where  $\Delta$  is the Hodge–Laplacian of differential forms. If  $C \cap \operatorname{int}\{\nabla u_i = 0\} \neq \emptyset$ , then by the unique continuation principle [Kaz88]  $\nabla u_i$  vanishes identically on  $C$ . Therefore  $C \cap \{\nabla u_i \neq 0\} = \emptyset$ , and  $C \cap \operatorname{cl}\{\nabla u_i \neq 0\} = \emptyset$  because  $C$  is open.

(b) As  $W$  is compactly contained in  $U$  we have  $W \subset \cup_{j=1}^N U_j$ , where the  $U_1, \dots, U_N$  are connected components of  $U$  that intersect  $\operatorname{spt}\|V\|$ . As  $U_j \cap \operatorname{spt}\|V\| \neq \emptyset$ , the weak convergence  $\|V^i\| \rightharpoonup \|V\|$  implies that there is  $I(j) \in \mathbb{N}$  such that  $U_j \cap \operatorname{spt}\|V^i\| \neq \emptyset$  for all  $i \geq I(j)$ . By (a) in fact  $U_j \subset \operatorname{spt}\|V^i\|$  for  $i \geq I(j)$ . Taking  $i \geq I(W) = \max_{j=1, \dots, N} I(j)$  yields  $W \subset \operatorname{spt}\|V^i\|$ .

(c) Consider a linear combination  $\phi \in \operatorname{span}(\phi_1, \dots, \phi_p)$  with  $|\nabla u_i|\phi = 0$ . Then  $0 = \frac{\epsilon_i}{2} \|\nabla u_i|\phi\|_{L^2}^2 = \|\phi\|_{L^2(V^i)}^2$ , so that  $\phi = 0$  in  $W \subset \operatorname{spt}\|V^i\|$  provided  $i \geq I(W)$ .  $\square$

We conclude the section with a proof of Theorem A(i) in the case where there is a uniform  $L^2(V^i)$ –bound on the second fundamental forms  $(A^i)$ .

**Lemma 3.13.** *Let  $W \subset\subset U \setminus \text{sing } V$  be open with  $W \cap \text{reg } V \neq \emptyset$ . If  $\sup_i \int_W |A^i|^2 d\|V^i\| < +\infty$ , then  $\lambda_p(W) \geq \limsup_{i \rightarrow \infty} \lambda_p^i(W)$  for all  $p$ .*

*Proof.* We may assume that every connected component of  $W$  intersects  $\text{spt}\|V\|$  (or  $\text{reg } V$ , equivalently as  $W \cap \text{sing } V = \emptyset$ ) without restricting generality: if  $C$  is a connected component of  $W$  with  $C \cap \text{reg } V = \emptyset$ , then  $\lambda_p(W \setminus C) = \lambda_p(W)$ , and for all  $i \in \mathbb{N}$  by monotonicity  $\lambda_p^i(W \setminus C) \geq \lambda_p^i(W)$ .

Given  $\delta > 0$  there is a  $p$ -dimensional linear subspace  $S = \text{span}(\phi_1, \dots, \phi_p) \subset H_0^1$  with

$$(3.30) \quad \lambda_p(W) + \delta \geq \max_{S \setminus \{0\}} J_V.$$

Extend the functions to  $C_c^1(W)$  as in Lemma 3.10. By Lemma 3.12 the functions  $(|\nabla u_i| \phi_1, \dots, |\nabla u_i| \phi_p)$  span a vector space  $|\nabla u_i|S \subset H_0^1(W)$  with  $\dim |\nabla u_i|S = p$  provided  $i \geq I(W)$ .

For all  $i \geq I(W)$ , let  $t_i \in \mathbb{S}^{p-1} \subset \mathbb{R}^p$  be the (normalised) coefficients of a linear combination  $t_i \cdot \phi = \sum_{j=1}^p t_{ij} \phi_j$  that maximises the Rayleigh quotient  $J_i$ :

$$(3.31) \quad J_i(t_i \cdot \phi) = \max_{S \setminus \{0\}} J_i \geq \lambda_p^i(W).$$

Extract a convergent subsequence  $t_{i'} \rightarrow t \in \mathbb{S}^{p-1}$ , so that  $t_{i'} \cdot \phi \rightarrow t \cdot \phi \in S$  in  $C_c^1(W)$  as  $i' \rightarrow \infty$ . Lemma 3.11 gives  $J_V(t \cdot \phi) \geq \limsup_{i \rightarrow \infty} \max_{S \setminus \{0\}} J_i$ , which in turn is greater than  $\limsup_{i \rightarrow \infty} \lambda_p^i(W)$ . By (3.30) and because  $\max_{S \setminus \{0\}} J_V \geq J_V(t \cdot \phi)$ ,

$$(3.32) \quad \lambda_p(W) + \delta \geq J_V(t \cdot \phi) \geq \limsup_{i \rightarrow \infty} \lambda_p^i(W),$$

and we conclude the proof of the lemma by letting  $\delta \rightarrow 0$ .  $\square$

Lemma 3.13 has the following immediate corollary.

**Corollary 3.14.** *Under the hypotheses of Lemma 3.13,*

$$(3.33) \quad \text{ind}_W B_V \leq \liminf_{i \rightarrow \infty} (\text{ind}_W u_i).$$

#### 4. PROOF OF THE MAIN THEOREM (THEOREM A)

We briefly recall the context of the proof:  $M^{n+1}$  is a closed Riemannian manifold and  $U \subset M$  is an arbitrary open subset. The sequence of functions  $(u_i)$  in  $C^3(U)$  satisfies Hypotheses (A), (B) and (C)—the last hypothesis says that  $\text{ind } u_i \leq k$  for all  $i$ . To every  $u_i$  we associate the  $n$ -varifold  $V^i$  from Definition 2.11. By Theorem 2.13, we may pass to a subsequence of  $(V^i)$  that converges weakly to a stationary integral varifold  $V$ .

**4.1. Spectrum and index of  $V$ : proof of (i) and (ii).** The main idea, inspired by an argument in [BW], is to fix an open subset  $W \subset\subset U \setminus \text{sing } V$  and study the stability of  $u_i$  in open balls covering  $W \cap \text{reg } V \neq \emptyset$ . We then shrink the radii of the covering balls to 0, and prove the spectral lower bound of Theorem A(i) by induction on  $k$ . The upper bound on  $\text{ind } B_V$  of Theorem A(ii) is then an immediate consequence.

In the base of induction the  $u_i$  are stable in  $U$ . Let  $\eta \in C_c^1(U)$  be a cutoff function with  $\eta|_W \equiv 1$ . The stability inequality (3.8) gives that

$$(4.1) \quad \int_W |A^i|^2 d\|V^i\| \leq \frac{C(M)}{\text{dist}(W, \partial U)^2} \|V^i\|(U) \quad \text{for all } i.$$

Combining this with (2.15) we get  $\sup_i \int_W |A^i|^2 d\|V^i\| < +\infty$ , and thus  $\lambda_p(W) \geq \limsup_{i \rightarrow \infty} \lambda_p^i(W)$  by Lemma 3.13.

For the induction step, let  $k \geq 1$  and assume that Theorem A(i) holds with  $k-1$  in place of  $k$ . Consider an arbitrary  $W \subset \subset U \setminus \text{sing } V$  that intersects  $\text{reg } V$ . Fix a radius  $0 < r < \text{dist}(W, \text{sing } V)$ , and pick points  $x_1, \dots, x_N \in W \cap \text{reg } V$  such that  $W \cap \text{reg } V \subset \cup_{j=1}^N B(x_j, \frac{r}{2})$ . We define the following *Stability Condition* for the cover  $\{B(x_j, \frac{r}{2})\}_{1 \leq j \leq N}$ :

(SC) For large  $i$ , each  $u_i$  is stable in every ball  $B(x_1, r), \dots, B(x_N, r)$ .

*Claim 1.* If for the cover  $\{B(x_j, \frac{r}{2})\}_{1 \leq j \leq N}$ :

- (a) (SC) holds, then  $\lambda_p(W) \geq \limsup_{i \rightarrow \infty} \lambda_p^i(W)$ .
- (b) (SC) fails, then  $\lambda_p(W \setminus \overline{B}) \geq \limsup_{i \rightarrow \infty} \lambda_p^i(W \setminus \overline{B})$  for some ball  $B \in \{B(x_j, r)\}$ .

*Proof.* (a) Let  $W_r = W \cap \cup_{j=1}^N B(x_j, \frac{r}{2})$ , so that  $W_r \cap \text{reg } V = W \cap \text{reg } V$  and hence  $\lambda_p(W_r) = \lambda_p(W)$ . Moreover  $W_r \subset W$ , so  $\lambda_p^i(W_r) \geq \lambda_p^i(W)$  for all  $i$  by monotonicity of the spectrum.

Because (SC) holds, summing (3.9) over all balls we get

$$(4.2) \quad \int_{W_r} |A^i|^2 d\|V^i\| \leq \frac{NC}{r^2} \|V^i\|(W_r) \leq \frac{NCE_0}{r^2\sigma} \quad \text{for all } i,$$

so that  $\lambda_p(W_r) \geq \limsup_{i \rightarrow \infty} \lambda_p^i(W_r)$  by Lemma 3.13, whence the conclusion follows.

(b) If (SC) fails, then some subsequence  $(u_{i'})$  must be unstable in a ball  $B$  of the cover, in other words  $\text{ind}_B u_{i'} \geq 1$  for all  $i'$ . On the other hand

$$(4.3) \quad \text{ind}_B u_{i'} + \text{ind}_{W \setminus \overline{B}} u_{i'} \leq \text{ind}_W u_{i'}$$

because  $B$  and  $W \setminus \overline{B}$  are disjoint open sets. Combining these with  $\text{ind}_W u_{i'} \leq k$ , we get  $\text{ind}_{W \setminus \overline{B}} u_{i'} \leq k-1$  for all  $i'$ , and we conclude after applying the induction hypothesis to  $(u_{i'})$  in  $W \setminus \overline{B}$ .  $\square$

*Remark 4.1.* Our argument shows that when (SC) fails there is a ball  $B \in \{B(x_j, r)\}$  with  $\lambda_p(W \setminus \overline{B}) \geq \limsup_{i \rightarrow \infty} \lambda_{p+1}^i(W)$  for  $p \geq k$ , and thus also that  $\text{ind}_{W \setminus \overline{B}} \leq k-1$ , but we only require the conclusion from Claim 1 for the induction step.

Consider a decreasing sequence  $r_m \rightarrow 0$  with  $0 < r_m < \text{dist}(W, \text{sing } V)$ . For each  $m$ , pick points  $x_1^m, \dots, x_{N_m}^m \in W \cap \text{reg } V$  such that

$$(4.4) \quad W \cap \text{reg } V \subset \cup_{j=1}^{N_m} B(x_j^m, \frac{r_m}{2}).$$

If (SC) holds for the cover  $\{B(x_j^m, \frac{r_m}{2})\}$  at some rank  $m$ , then  $\lambda_p(W) \geq \limsup_{i \rightarrow \infty} \lambda_p^i(W)$  by Claim 1, and the induction step is completed.



Otherwise (SC) fails for all constructed covers, and by Claim 1 there is a sequence  $(y_m)$  in  $W \cap \text{reg } V$  with

$$(4.5) \quad \lambda_p(W \setminus \overline{B}(y_m, r_m)) \geq \limsup_{i \rightarrow \infty} \lambda_p^i(W \setminus \overline{B}(y_m, r_m)).$$

Using the monotonicity of the spectrum on the right-hand side, we get

$$(4.6) \quad \lambda_p(W \setminus \overline{B}(y_m, r_m)) \geq \limsup_{i \rightarrow \infty} \lambda_p^i(W).$$

Passing to a subsequence if necessary, we may assume that  $(y_m)$  converges to a point  $y \in \overline{W} \cap \text{reg } V$ . If we fix a radius  $R > 0$ , then  $B(y_m, r_m) \subset B(y, R)$  for large enough  $m$ , so by monotonicity and (4.6):

$$(4.7) \quad \begin{aligned} \lambda_p(W \setminus \overline{B}(y, R)) &\geq \limsup_{m \rightarrow \infty} \lambda_p(W \setminus \overline{B}(y_m, r_m)) \\ &\geq \limsup_{i \rightarrow \infty} \lambda_p^i(W). \end{aligned}$$

The conclusion follows after combining this with  $\lambda_p(W) = \lim_{R \rightarrow 0} \lambda_p(W \setminus \overline{B}(y, R))$  from Lemma 3.7.

Together with the base of induction, we have proved Theorem A(i) for all sequences  $(u_i)$  with  $\sup_i \text{ind } u_i \leq k$  for some  $k \in \mathbb{N}$ . The index bound  $\text{ind}_W B_V \leq k$  follows immediately: the spectral lower bound implies that  $L_V$  must have fewer negative eigenvalues than the  $L_i$  as  $i \rightarrow \infty$ . Therefore

$$(4.8) \quad \text{ind}_W B_V = \text{card}\{p \in \mathbb{N} \mid \lambda_p(W) < 0\} \leq k.$$

As the subset  $W$  was arbitrary we get  $\text{ind } B_V \leq k$ ; this proves Theorem A(ii).

**4.2. Regularity of  $V$ : proof of  $\dim_{\mathcal{H}} \text{sing } V \leq n - 7$ .** The approach is the same as in the proof of Theorem A(i), with one difference: we proceed by induction on  $k$ , but we now cover the entire support  $\text{spt}\|V\|$  (including the singular set), instead of constructing covers a positive distance away from  $\text{sing } V$ .

The base of induction, where the  $u_i$  are stable in  $U$ , was proved in [TW12].

For the induction step, suppose that  $\dim_{\mathcal{H}}(U' \cap \text{sing } V) \leq n - 7$  holds with  $k - 1$  in place of  $k$ , and for arbitrary open subsets  $U' \subset U$ . Fix  $r > 0$ , and choose points  $x_1, \dots, x_N \in U \cap \text{spt}\|V\|$  such that  $U \cap \text{spt}\|V\| \subset \cup_{j=1}^N B(x_j, r)$ . The *Stability Condition* for the cover  $\{B(x_j, r)\}_{1 \leq j \leq N}$  is defined in almost the same way as before:

(SC) For large  $i$ , each  $u_i$  is stable in every ball  $B(x_1, r), \dots, B(x_N, r)$ .

*Claim 2.* If for the cover  $\{B(x_j, r)\}_{1 \leq j \leq N}$ :

(a) (SC) holds, then  $\dim_{\mathcal{H}} \text{sing } V \leq n - 7$ .

(b) (SC) fails, then  $\dim_{\mathcal{H}} \text{sing } V \setminus \overline{B} \leq n - 7$  for some ball  $B \in \{B(x_j, r)\}$ .

*Proof.* (a) The results from [TW12] give  $\dim_{\mathcal{H}} B(x_j, r) \cap \text{sing } V \leq n - 7$  for every  $j = 1, \dots, N$ . As the balls  $\{B(x_j, r)\}$  cover  $U \cap \text{spt}\|V\|$ , the same holds for  $\text{sing } V$ .

(b) Because (SC) fails, there must be a subsequence  $(u_{i'})$  that is unstable in one of the balls  $B$  of the cover, so that in its complement

$$(4.9) \quad \text{ind}_{U \setminus \overline{B}} u_{i'} \leq k - 1 \quad \text{for all } i'.$$

The conclusion follows from the induction hypothesis applied to  $(u_{i'})$  in  $U \setminus \overline{B}$ .  $\square$

Now consider a decreasing sequence  $r_m \rightarrow 0$ . For every  $m$ , choose points  $x_1^m, \dots, x_{N_m}^m \in U \cap \text{spt}\|V\|$  such that  $U \cap \text{spt}\|V\| \subset \bigcup_{j=1}^{N_m} B(x_j^m, r_m)$ . Then either (SC) holds for the cover  $\{B(x_j^m, r_m)\}$  constructed for some  $m$ , in which case we can conclude from Claim 2, or else there is sequence of points  $(y_m)$  in  $U \cap \text{spt}\|V\|$  for which

$$(4.10) \quad \dim_{\mathcal{H}} \text{sing } V \setminus \overline{B}(y_m, r_m) \leq n - 7.$$

Possibly after extracting a subsequence, the sequence  $(y_m)$  converges to a point  $y \in \overline{U} \cap \text{spt}\|V\|$ . As  $U \setminus \{y\} \subset \bigcup_{m \geq 0} U \setminus \overline{B}(y_m, r_m)$ , we get  $\dim_{\mathcal{H}}(\text{sing } V \setminus \{y\}) \leq n - 7$ .

If  $n \geq 7$ , then  $\dim_{\mathcal{H}} \text{sing } V \leq n - 7$  holds whether or not  $y \in \text{sing } V$ , as points are zero-dimensional. If however  $2 \leq n \leq 6$  then we need  $\text{sing } V = \emptyset$ , which amounts to the following claim.

*Claim 3.* If  $2 \leq n \leq 6$  then  $y \notin \text{sing } V$ .

*Proof.* Choose  $B(y, R) \subset U$ , and consider balls  $\{B(y, R_m)\}_{m \in \mathbb{N}}$  with shrinking radii  $R_m := 2^{-m}R$ . If for some  $m$  there is a subsequence  $(u_{i'})$  with

$$(4.11) \quad \text{ind}_{B(y, R_m)} u_{i'} \leq k - 1 \quad \text{for all } i',$$

then we can conclude from the induction hypothesis. Otherwise for all  $m$

$$(4.12) \quad \text{ind}_{B(y, R_m)} u_i = k \quad \text{for } i \text{ large enough,}$$

and the  $u_i$  are eventually stable in the annulus  $B(y, R) \setminus \overline{B}(y, R_m)$ . By Theorem A(ii)

$$(4.13) \quad \text{ind}_{B(y, R) \setminus \overline{B}(y, R_m)} B_V = 0 \quad \text{for all } R_m \rightarrow 0,$$

and thus  $\text{ind}_{B(y, R) \setminus \{y\}} B_V = 0$ .

By contradiction, suppose that  $y \in \text{sing } V$ . Then  $\text{ind}_{B(y, r)} B_V = 0$  holds in the whole ball  $B(y, r)$  away from  $\text{sing } V$ , and the regularity results from [Wic14] give  $\dim_{\mathcal{H}} B(y, R) \cap \text{sing } V \leq n - 7$ , so that  $y \notin \text{sing } V$ .  $\square$

Claim 3 concludes the induction step; together with the base of induction, we have proved that  $\dim_{\mathcal{H}} \text{sing } V \leq n - 7$ . This finishes the proof of Theorem A.

## APPENDIX A. MEASURE-FUNCTION CONVERGENCE

In this appendix we give two simple abstract measure-theoretical lemmas that are used in the proofs of Lemma 3.11 and in Appendix B. Essentially they give a weak compactness result for sequences of the form  $(f_i \, d\mu_i)_{i \in \mathbb{N}}$ , with  $\mu_i$  Radon measures and  $f_i \in L^2(\mu_i)$ . The weak convergence in question is sometimes called *measure-function convergence* in the literature. It appears in [Hut86] in the context of so-called curvature varifolds; there one also finds a proof of Lemma A.1 under more general hypotheses on  $(f_i)$ .

**Lemma A.1** ([Hut86, Ton05]). *Let  $X$  be a locally compact Hausdorff space, let  $(\mu_i)_{i \in \mathbb{N}}$  be a sequence of Radon measures on  $X$ , and  $(f_i : X \rightarrow \mathbb{R})_{i \in \mathbb{N}}$  be*

a sequence of Borel-measurable functions. Suppose that

$$(A.1) \quad \sup_i \mu_i(X) < +\infty,$$

$$(A.2) \quad \sup_i \int_X f_i^2 d\mu_i < +\infty.$$

Then there is a Radon measure  $\mu$  and  $f \in L^2(\mu)$  such that for some subsequence  $\mu_{i'} \rightarrow \mu$  and  $f_{i'} d\mu_{i'} \rightarrow f d\mu$  weakly as Radon measures, i.e.

$$(A.3) \quad \int_X f_{i'} \phi d\mu_{i'} \rightarrow \int_X f \phi d\mu \quad \text{for all } \phi \in C_c(X).$$

Moreover, the weak limit  $f d\mu$  satisfies

$$(A.4) \quad \int_X f^2 d\mu \leq \liminf_{i \rightarrow \infty} \int_X f_i^2 d\mu_i.$$

*Remark A.2.* In our applications  $X$  is either an open subset of  $U \subset M$  or its Grassmannian  $G_n(U)$ , and  $\mu_i$  is either  $\|V^i\|$  or  $V^i$ .

*Proof.* The signed Radon measure  $\nu_i := f_i d\mu_i$  has bounded total variation, so that the sequences  $(\mu_i)$  and  $(\nu_i)$  have convergent subsequences, with limits the Radon measures  $\mu$  and  $\nu$  respectively. Extract these subsequences without relabelling their indices.

Consider an arbitrary  $\phi \in C_c(X)$ . By the weak convergence  $\nu_i \rightarrow \nu$ ,

$$(A.5) \quad \int \phi d\nu = \lim_{i \rightarrow \infty} \int \phi f_i d\mu_i \leq \|\phi\|_{L^2(\mu)} \liminf_{i \rightarrow \infty} \|f_i\|_{L^2(\mu_i)},$$

where we used the weak convergence  $\mu_i \rightarrow \mu$  to conclude  $\lim_{i \rightarrow \infty} \|\phi\|_{L^2(\mu_i)} = \|\phi\|_{L^2(\mu)}$ . As  $C_c(X)$  is dense in  $L^2(X)$ , the measure  $\nu$  defines a bounded linear functional on  $L^2(X)$ , and by duality there is  $f \in L^2(\mu)$  with  $\|f\|_{L^2(\mu)} \leq \liminf_{i \rightarrow \infty} \|f_i\|_{L^2(\mu_i)}$  such that  $\nu = f d\mu$ .  $\square$

If the densities  $f_i$  are in  $C_c(X)$  and converge strongly, then their limit coincides with the density of the weak limit of  $(f_i d\mu_i)$ .

**Corollary A.3.** *Additionally to the hypotheses of Lemma A.1, assume that  $f_i \in C_c(X)$ , and that  $\|f_i - f\|_{L^\infty} \rightarrow 0$  for some  $f \in C_c(X)$ . Then, additionally to the conclusions of Lemma A.1:*

$$(A.6) \quad \int_X f^2 d\mu = \lim_{i \rightarrow \infty} \int_X f_i^2 d\mu_i.$$

*Proof.* We first show that  $f_i d\mu_i \rightarrow f d\mu$ . Let  $\varphi \in C_c(X)$  be arbitrary, then

$$(A.7) \quad \int f_i \varphi d\mu_i - \int f \varphi d\mu = \int (f_i - f) \varphi d\mu_i + \int f \varphi d\mu_i - \int f \varphi d\mu.$$

The first term  $|\int (f_i - f) \varphi d\mu_i| \leq \|f_i - f\|_{L^\infty} \|\varphi\|_{L^1(\mu_i)} \rightarrow 0$  as  $i \rightarrow \infty$ . The remaining terms  $\int f \varphi d\mu_i - \int f \varphi d\mu \rightarrow 0$  as  $i \rightarrow \infty$  by the weak convergence  $\mu_i \rightarrow \mu$ . We reason similarly to show (A.6):

$$(A.8) \quad \left| \int_X f^2 d\mu - \int_X f_i^2 d\mu_i \right| \leq \left| \int_X f^2 d\mu - \int_X f^2 d\mu_i \right| + \|f_i^2 - f^2\|_{L^\infty} \mu_i(X).$$

The first term goes to 0 by the weak convergence  $\mu_i \rightarrow \mu$ , and so does the second as  $\sup_i \mu_i(X) < +\infty$  and  $\|f_i^2 - f^2\|_{L^\infty} \rightarrow 0$ .  $\square$

## APPENDIX B. GENERALISED SECOND FUNDAMENTAL FORMS

Our main aim in this appendix is to give a proof of Proposition 3.6. We follow the approach of [Ton05], where the case of stable  $u_i$  is treated using notions from [Hut86]. Our account is self-contained but for the fact that we refer to these two works for some technical, but routine calculations.

Throughout this section, we assume that  $U \subset M$  is isometrically embedded in some  $\mathbb{R}^q$ , and  $W \subset\subset U \setminus \text{sing } V$  is an open subset with  $W \cap \text{reg } V \neq \emptyset$ . The fibre of the Grassmannian  $G_n(U)$  at  $x \in U$  is identified with

$$(B.1) \quad \{S \subset \mathbb{R}^q \mid S \subset T_x M, \dim S = n\} \subset U \times G(n, q),$$

where  $G(n, q) = \{S \subset \mathbb{R}^q \mid \dim S = n\}$ . We furthermore identify an element  $S \in U \times G(n, q)$  with the corresponding orthogonal projection  $\mathbb{R}^q \rightarrow S$ , so that  $G_n(U) \subset U \times \mathbb{R}^{q^2}$ . Throughout,  $P(x) \in \mathbb{R}^{q^2}$  represents the orthogonal projection  $\mathbb{R}^q \rightarrow T_x M$  and  $(e_1, \dots, e_q)$  is the standard basis of  $\mathbb{R}^q$ ;  $\partial_i$  and  $\partial_{ij}^*$  denote differentiation with respect to  $e_i$  and  $e_i \otimes e_j \in \mathbb{R}^{q^2}$  respectively.

Start by considering a smooth embedded hypersurface  $\Sigma \subset U$ , which we implicitly identify with the varifold  $V_\Sigma := V_{\Sigma,1}$  with constant multiplicity. Let  $\phi \in C^1(G_n(U))$  be a scalar test function with compact spatial support, and associate to it  $\varphi \in C_c^1(\Sigma)$  defined by  $\varphi(x) = \phi(x, S^\Sigma(x))$ , where  $S^\Sigma(x) \in \mathbb{R}^{q^2}$  represents the orthogonal projection  $\mathbb{R}^q \rightarrow T_x \Sigma$ . Define a vector field  $X \in C_c^1(\Sigma, TM)$  by  $X = \varphi P(e_j)$ , where  $e_j$  is one of the standard basis vectors. Its component tangent to  $\Sigma$  is  $\varphi S^\Sigma(e_j)$ , and by the standard divergence theorem we get  $\int_\Sigma \text{div}_\Sigma(\varphi S^\Sigma e_j) = 0$ . A routine calculation shows that in coordinates

$$(B.2) \quad \text{div}_\Sigma(\varphi S^\Sigma e_j) = S_{rj}^\Sigma \partial_r \phi + S_{ri}^\Sigma \partial_i S_{jr}^\Sigma + S_{ji}^\Sigma \partial_i S_{kr}^\Sigma \partial_{kr}^* \phi,$$

with summation over repeated indices [Hut86]. Abbreviate  $B_{jkr}^\Sigma = S_{ji}^\Sigma \partial_i S_{kr}^\Sigma$  and substitute this into the divergence formula:

$$(B.3) \quad 0 = \int_\Sigma S_{rj}^\Sigma \partial_r \phi + B_{rjr}^\Sigma \phi + B_{jkr}^\Sigma \partial_{kr}^* \phi \, d\mathcal{H}^n$$

$$(B.4) \quad = \int_{G_n(U)} S_{rj} \partial_r \phi + B_{rjr}^\Sigma \phi + B_{jkr}^\Sigma \partial_{kr}^* \phi \, dV_\Sigma(x, S).$$

This identity is the basis of the following definition.

**Definition B.1** (Generalised curvature, [Hut86]). An  $n$ -dimensional integral varifold  $V$  in  $U$  is said to have *generalised curvature* if there exists a function  $B = (B_{ijk})$  with values in  $\mathbb{R}^{q^3}$  defined  $V$ -a.e. on  $G_n(U)$  with

- (a)  $B \in L_{\text{loc}}^1(V)$ ,
- (b)  $\int_{G_n(U)} S_{rj} \partial_r \phi + B_{rjr}^\Sigma \phi + B_{jkr}^\Sigma \partial_{kr}^* \phi \, dV(x, S) = 0$  for all  $\phi \in C^1(G_n(U))$  with compact spatial support.

The following lemma shows that the function  $B$  is well-defined  $V$ -a.e. on  $G_n(U)$ ; its proof is taken verbatim from [Hut86].

**Lemma B.2.** Any two  $B$  and  $\tilde{B}$  satisfying (a) and (b) coincide  $V$ -a.e. on  $G_n(U)$ .

*Proof.* Let  $\phi(x, S) = \alpha(x)\beta(S)$ , where  $\alpha \in C_c^1(U)$  and  $\beta \in C^1(\mathbb{R}^{q^2})$ . Letting  $\beta \equiv 1$  we see that  $\int B_{rjr}\alpha dV = \int \tilde{B}_{rjr}\alpha dV$ , and thus  $B_{rjr} = \tilde{B}_{rjr}$   $V$ -a.e. on  $G_n(U)$ . If we now let

$$(B.5) \quad \beta(S) = \begin{cases} 1 & \text{if } S = S_{kr} \\ 0 & \text{otherwise} \end{cases}$$

then  $\int B_{jkr}\alpha dV = \int \tilde{B}_{jkr}\alpha dV$ , whence the conclusion follows.  $\square$

In particular, applied to the smooth hypersurfaces, the following is an immediate consequence.

**Corollary B.3.** *If  $\Sigma$  is a smoothly embedded hypersurface, then  $B_{ijk}(x, S) = B_{ijk}^\Sigma(x, S)$  for  $V_\Sigma$ -a.e.  $(x, S) \in G_n(U)$ , where  $B_{ijk}^\Sigma(x, S) = S_{il}\partial_l S_{jk}^\Sigma$ .*

The following elementary calculation relates  $B^\Sigma$  to the second fundamental form  $A^\Sigma$ .

**Lemma B.4.** *Let  $A^\Sigma$  be the second fundamental form of a smoothly embedded hypersurface  $\Sigma \subset U$ . Then*

$$(B.6) \quad \langle A^\Sigma(S^\Sigma e_i, S^\Sigma e_j), Pe_k \rangle = P_{kr} S_{js}^\Sigma S_{il}^\Sigma \partial_l S_{rs}^\Sigma = P_{kr} S_{js}^\Sigma B_{irs}^\Sigma(x, S^\Sigma).$$

*Proof.* Write  $A$  instead of  $A^\Sigma$  in this proof to simplify notation. The co-variant derivative on  $M$  is the component of  $D = \nabla^{\mathbb{R}^q}$  tangent to  $M$ , so  $A = (D^{T_M})^\perp = (D^\perp)^{T_M}$ . As  $e_k^{T_M} = P_{kr} e_r$ , we get:

$$(B.7) \quad \begin{aligned} A_{ij}^k &:= \langle A(S^\Sigma e_i, S^\Sigma e_j), Pe_k \rangle \\ &= \langle (D_{S^\Sigma e_i} S^\Sigma e_j)^\perp, e_k^{T_M} \rangle = P_{kr} \langle D_{S^\Sigma e_i} S^\Sigma e_j, e_r^\perp \rangle. \end{aligned}$$

Similarly  $e_r^\perp = (\delta_{rs} - S_{rs}^\Sigma) e_s$ , so:

$$(B.8) \quad A_{ij}^k = P_{kr} (\delta_{rs} - S_{rs}^\Sigma) \langle D_{S^\Sigma e_i} S^\Sigma e_j, e_s \rangle = P_{kr} (\delta_{rs} - S_{rs}^\Sigma) D_{S^\Sigma e_i} S_{js}^\Sigma.$$

As  $S_{rs}^\Sigma S_{js}^\Sigma = S_{rj}^\Sigma$ , we finally get  $A_{ij}^k = P_{kr} S_{js}^\Sigma D_{S^\Sigma e_i} S_{rs}^\Sigma = P_{kr} S_{js}^\Sigma S_{il}^\Sigma \partial_l S_{rs}^\Sigma$ , as required.  $\square$

We use this expression to generalise second fundamental forms from the smooth to the varifold setting.

**Definition B.5** (Generalised second fundamental forms, [Hut86]). Let  $V$  be an integral  $n$ -varifold with generalised curvature. Then its *generalised second fundamental form* is the function  $A = (A_{ij}^k)$  with values in  $\mathbb{R}^{q^3}$  and defined at  $V$ -a.e.  $(x, S) \in G_n(U)$  by

$$(B.9) \quad A_{ij}^k(x, S) = P_{kr} S_{js} B_{irs}.$$

For a smoothly embedded  $\Sigma \subset U$ , we see after combining Corollary B.3 with Lemma B.4 that the generalised second fundamental form  $A$  of  $V_\Sigma$  is equal to the classical second fundamental form  $A^\Sigma$  in the sense that

$$(B.10) \quad A_{ij}^k(x, S) = \langle A^\Sigma(S e_i, S e_j), P e_k \rangle \quad \text{for } V_\Sigma\text{-a.e. } (x, S) \in G_n(U).$$

We now want to relate these notions to the varifolds  $V^i$  defined in the main body. To simplify notation, let us fix a  $u = u_i$  with associated varifold  $V^\epsilon = V^i$ . We define a ‘second fundamental form’ for  $V^\epsilon$  using the coordinate expressions from Lemma B.4.

**Definition B.6.** Define the functions  $A^\epsilon = (A_{ij}^{\epsilon,k})$  and  $B^\epsilon = (B_{ijk}^\epsilon)$  with values in  $\mathbb{R}^{q^3}$  at all  $(x, S) \in G_n(U)$  where  $\nabla u(x) \neq 0$  by

$$(B.11) \quad B_{ijk}^\epsilon(x, S) = S_{il} \partial_l S_{jk}^\epsilon,$$

$$(B.12) \quad A_{ij}^{k,\epsilon}(x, S) = P_{kr} S_{js} S_{il} \partial_l S_{rs}^\epsilon = P_{kr} S_{js} B_{irs}^\epsilon,$$

where  $S^\epsilon = S^\epsilon(x) \in \mathbb{R}^{q^2}$  represents the projection  $\mathbb{R}^q \rightarrow T_x\{u = u(x)\}$ , and  $P = P(x) \in \mathbb{R}^{q^2}$  the projection  $\mathbb{R}^q \rightarrow T_x M$ .

Technically speaking the function  $A^\epsilon$  is not the second fundamental form of  $V^\epsilon$ , as  $B^\epsilon$  satisfies the integral identity of Definition B.6 only up to a small error term. This can be seen as follows: for arbitrary  $X \in C_c^1(U, TM)$ , multiply the Allen–Cahn equation by  $X \cdot \nabla u$  and integrate by parts twice to obtain

$$(B.13) \quad \int_U |\nabla u|^2 \operatorname{div} X - \langle \nabla_{\nabla u} X, \nabla u \rangle = \int_U \left( \frac{|\nabla u|^2}{2} - \frac{W(u)}{\epsilon^2} \right) \operatorname{div} X,$$

which using integration with respect to  $V^\epsilon$  is equivalent to

$$(B.14) \quad \int_{G_n(U)} \operatorname{div}_S X \, dV^\epsilon(x, S) = \frac{1}{2\sigma} \int_U \left( \epsilon \frac{|\nabla u|^2}{2} - \frac{W(u)}{\epsilon} \right) \operatorname{div} X,$$

where  $\operatorname{div}_S X = \sum_{i=1}^q \langle D_{Se_i} X, Se_i \rangle$ . As before let  $X = \phi(x, S^\epsilon) S^\epsilon(e_j)$ , where  $\phi \in C^1(G_n(U))$  has compact spatial support. Substitute this into (B.14) and repeat the routine computations alluded to before (B.2) to get

$$(B.15) \quad \int_{G_n(U)} S_{rj} \partial_r \phi + B_{rjr}^\epsilon \phi + B_{jkr}^\epsilon \partial_{kr}^* \phi \, dV^\epsilon(x, S) = \frac{1}{2\sigma} \int_U \left( \epsilon \frac{|\nabla u|^2}{2} - \frac{W(u)}{\epsilon} \right) \operatorname{div} X.$$

The integral on the right-hand side goes to 0 as  $\epsilon \rightarrow 0$ —this is (2.16) in Theorem 2.13. This justifies the abuse of language that is calling  $A^\epsilon$  the ‘second fundamental form’ of  $V^\epsilon$ .

Suppose that  $\nabla u(x) \neq 0$  at some point  $x \in U$ . Then the level set  $\{u = u(x)\}$  is embedded in a neighbourhood  $B$  say of  $x$ . Write  $\Sigma = \{u = u(x)\} \cap B$  and notice that the calculations from Lemma B.4 show that  $A^\epsilon(x, S^\epsilon) = A^\Sigma(x)$ , so the second fundamental forms from Definitions 3.1 and B.6 agree  $V^\epsilon$ -a.e. Combining this observation with (3.2), we get

$$(B.16) \quad |A^\epsilon|^2(x, S^\epsilon) = |A^\Sigma|^2(x) \leq \frac{1}{|\nabla u|^2} (|\nabla^2 u|^2 - |\nabla |\nabla u||^2).$$

Therefore, when  $\delta^2 E_\epsilon(u)(|\nabla u| \phi, |\nabla u| \phi) \geq 0$  for some  $\phi \in H_0^1(U)$ , then  $\int_{G_n(U)} |A^\epsilon|^2 \phi^2 \, dV^\epsilon \leq \int_U |\nabla \phi|^2 - \operatorname{Ric}(\nu^\epsilon, \nu^\epsilon) \phi^2 \, d\|V^\epsilon\|$  as in (3.8), and Corollary 3.5 also remains valid.

All the results in this appendix were established for an arbitrary critical point  $u \in C^3(U) \cap L^\infty(U)$ , and thus are valid for every term in the sequence  $(u_i)$  satisfying Hypotheses (A)–(C). Let  $(V^i)$  be the corresponding varifolds as in Definition 2.11, and let  $(A^{\epsilon_i})$  be their second fundamental forms as in Definition B.6. We restate Proposition 3.6 in the following equivalent form, with  $A^{\epsilon_i}$  in place of  $A^i$ .

**Proposition B.7.** *If  $\sup_i \int_W |A^{\epsilon_i}|^2 d\|V^i\| < +\infty$ , then some subsequence  $A^{\epsilon_{i'}} dV^{i'} \rightharpoonup A dV$  weakly as Radon measures on  $G_n(W)$ , and*

$$(B.17) \quad \int_W |A|^2 d\|V\| \leq \liminf_{i \rightarrow \infty} \int_W |A^{\epsilon_i}|^2 d\|V^i\|,$$

where  $A$  is the classical second fundamental form of  $\text{reg } V \subset M$ .

*Proof.* Routine calculations as above in the proof of Lemma B.4 show that  $B^{\epsilon_i}$  is related to  $A^{\epsilon_i}$  as follows for all  $i$ :

$$(B.18) \quad B_{jkl}^{\epsilon_i}(x, S^{\epsilon_i}) = A_{jk}^{l, \epsilon_i} + A_{jl}^{k, \epsilon_i} + S_{ks}^{\epsilon_i} S_{jr}^{\epsilon_i} \partial_r P_{sl} + S_{ls}^{\epsilon_i} S_{jr}^{\epsilon_i} \partial_r P_{ks}.$$

If we square (B.18) and sum over  $j, k, l = 1, \dots, q$ , we get

$$(B.19) \quad |B^{\epsilon_i}|^2 \leq 8(|A^{\epsilon_i}|^2 + |DP|^2) \quad V^i\text{-a.e. in } G_n(U)$$

The term  $|DP|^2 := \sum_{j,k,l}^q (\partial_j P_{kl})^2$  can be bounded by some constant  $C(M)$ , so that  $\sup_i \int_W |B^{\epsilon_i}|^2 d\|V^i\| < +\infty$  as well.

By Lemma A.1 we can pass to convergent subsequences  $A^{\epsilon_i} dV^i \rightharpoonup A dV$  and  $B^{\epsilon_i} dV^i \rightharpoonup B dV$  with limits related by  $A_{jk}^l = P_{lr} S_{ks} B_{jrs}$   $V$ -a.e. in  $G_n(W)$  and satisfy

$$(B.20) \quad \int_{G_n(W)} |A|^2 dV \leq \liminf_{i \rightarrow \infty} \int_{G_n(W)} |A^{\epsilon_i}|^2 dV^i,$$

as well as the same inequality for  $B$ . Moreover the error term on the right-hand side of (B.15) tends to 0 as  $i \rightarrow \infty$ , so the weak limit  $B dV$  satisfies

$$(B.21) \quad \int_{G_n(W)} S_{rj} \partial_r \phi + B_{rjr} \phi + B_{jkr} \partial_{kr}^* \phi dV(x, S) = 0$$

for all  $\phi \in C^1(G_n(W))$  with compact spatial support. By Corollary B.3 we have  $B = B^{\text{reg } V}$  and thus also  $A = A^{\text{reg } V}$   $V$ -a.e. in  $G_n(W)$ . This concludes the proof.  $\square$

## REFERENCES

- [ACS15] L. Ambrozio, A. Carlotto, and B. Sharp. Compactness of the space of minimal hypersurfaces with bounded volume and  $p$ -th Jacobi eigenvalue. *ArXiv e-prints*, May 2015.
- [Alm65] Frederick Almgren. *The theory of varifolds: A variational calculus in the large for the  $k$ -dimensional area integrand*. Institute of Advanced Study, 1965.
- [BW] Costante Bellettini and Neshan Wickramasekera. Regularity of stable codimension 1 CMC varifolds. Manuscript in preparation.
- [CM11] T.H. Colding and W.P. Minicozzi. *A Course in Minimal Surfaces*. Graduate studies in mathematics. American Mathematical Society, 2011.
- [DLT13] Camillo De Lellis and Dominik Tasnady. The existence of embedded minimal hypersurfaces. *J. Differential Geom.*, 95(3):355–388, 11 2013.
- [EG15] L.C. Evans and R.F. Gariepy. *Measure Theory and Fine Properties of Functions, Revised Edition*. Textbooks in Mathematics. CRC Press, 2015.
- [FSV13] Alberto Farina, Yannick Sire, and Enrico Valdinoci. Stable solutions of elliptic equations on Riemannian manifolds. *Journal of Geometric Analysis*, 23(3):1158–1172, 2013.
- [GT98] D. Gilbarg and N.S. Trudinger. *Elliptic partial differential equations of second order*. Grundlehren der mathematischen Wissenschaften. Springer, 1998.
- [Gua15] Marco Guaraco. Min-max for phase transitions and the existence of embedded minimal hypersurfaces. Preprint, 2015. <http://arxiv.org/abs/1505.06698>.

- [HT00] John Hutchinson and Yoshihiro Tonegawa. Convergence of phase interfaces in the van der Waals-Cahn-Hilliard theory. *Calculus of Variations and Partial Differential Equations*, 10(1):49–84, 2000.
- [Hut86] John Hutchinson. Second fundamental form for varifolds and the existence of surfaces minimising curvature. *Indiana University Mathematical Journal*, 35(1):45–71, 1986.
- [Ilm93] Tom Ilmanen. Convergence of the Allen-Cahn equation to Brakke’s motion by mean curvature. *J. Differential Geom.*, 38(2):417–461, 1993.
- [Kaz88] Jerry L Kazdan. Unique continuation in geometry. *Communications on Pure and Applied Mathematics*, 41(5):667–681, 1988.
- [MN15] Fernando Codá Marques and André Neves. Morse index and multiplicity of min-max minimal hypersurfaces. Preprint, 2015. <http://arxiv.org/abs/1512.06460>.
- [Pit81] Jon Pitts. *Existence and regularity of minimal surfaces on Riemannian manifolds*, volume 27 of *Mathematical Notes*. Princeton University Press, 1981.
- [Sim84] Leon Simon. *Lectures on geometric measure theory*. Proceedings of the Center for Mathematical Analysis. Australian National University, 1984.
- [SS81] Richard Schoen and Leon Simon. Regularity of stable minimal hypersurfaces. *Communications on Pure and Applied Mathematics*, 34(6):741–797, 1981.
- [SSY75] Richard Schoen, Leon Simon, and Shing-Tung Yau. Curvature estimates for minimal hypersurfaces. *Acta Mathematica*, 134(1):275–288, 1975.
- [Ton05] Yoshihiro Tonegawa. On stable critical points for a singular perturbation problem. *Communications in Analysis and Geometry*, 13(2):439–459, 2005.
- [TW12] Yoshihiro Tonegawa and Neshan Wickramasekera. Stable phase interfaces in the van der Waals–Cahn–Hilliard theory. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2012(668):191–210, 2012.
- [Wic14] Neshan Wickramasekera. A general regularity theory for stable codimension 1 integral varifolds. *Annals of Mathematics*, 179(3):843–1007, 2014.